Heat Flows on Dirichlet Domains: Exponential Relaxation and Normalized Gradient Dynamics

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Basic Theory

Partial Derivatives

$$\frac{\partial u}{\partial x_i} = \lim_{h\to 0} \frac{u(x + h\,e_i) - u(x)}{h}$$

Change along a coordinate axis

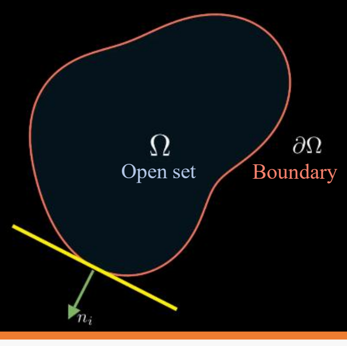
Gradient:

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right)$$

Laplace operator:

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$$

Open Set and Boundary



Function Spaces

The Space of Square Integrable Functions

$$L^2(\Omega) = \{f : \Omega \to \mathbb{R} \mid \int_\Omega |f(x)|^2\,dx < \infty\} \qquad \text{Norm: } \|f\| = \left(\int_\Omega |f(x)|^2 dx\right)^{1/2}$$

Sobolev Space

$$H_0^1(\Omega) = \left\{u \in L^2(\Omega) \mid \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ u|_{\partial\Omega} = 0\right\} \qquad \text{Norm: } \|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}$$

Two Main Results

Exponential Relaxation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f \text{ in } \Omega, \\ u(t,x) = 0 \text{ on } \partial\Omega, \\ u(0,x) = u_0(x) \end{cases} \quad (1) \qquad \begin{cases} -\Delta \tilde{u} = f \text{ in } \Omega, \\ \tilde{u} = 0 \text{ on } \partial\Omega \end{cases} \quad (2)$$

The solution u(t, x) of (1) converges exponentially to the steady-state solution of (2) as $t \to \infty$.

Normalized Gradient Flow

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \mu(t)\cdot u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0,x) = u_0(x), \text{and } \|u\|_{L^2(\Omega)} = 1. \end{cases} \quad (3)$$

There exists a unique solution $u_N(t)$ $\in E_N$ for every N, and u_N converges to a weak solution of (3) as $N \to \infty$.

Fundamental Theorems

Green's Formula

$$\int_\Omega (\nabla u \cdot \nabla v)\,dx = -\int_\Omega v\,\Delta u\,dx + \int_{\partial\Omega} v\,\frac{\partial u}{\partial n}\,ds$$

Fourier Series Expansion

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t)\,\phi_k(x)$$

ϕ_k are orthogonal basis functions: $\begin{cases}\langle \phi_i, \phi_j\rangle_{L^2} = 0 \text{ for } i \neq j, \\ \langle \phi_i, \phi_i\rangle_{L^2} = 1\end{cases}$

$$E_N = \text{span}\{v_1, v_2, \dots, v_N\} = \left\{f(x) = \sum_{i=1}^{N} c_i v_i(x),\ c_i \in \mathbb{R}\right\}$$

Proof Sketch of Main Results

Variational Formulation

$$\frac{\partial u}{\partial t} - \Delta u = f$$

Multiply by test function $v \in H_0^1(\Omega)$ → Integrate over Ω

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx - \int_\Omega (\Delta u) v\,dx = \int_\Omega f v\,dx$$

Green's formula: Since $v \in H_0^1(\Omega)$, holds $\int_\Omega (\nabla u \cdot \nabla v)\,dx = -\int_\Omega v\,\Delta u\,dx + 0$

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx, \text{ for all } v \in H_0^1(\Omega) \quad ①$$

Integrate over time $t \in [0, T]$

$$\int_\Omega u(t,x)\,v(x)\,dx + \int_0^t \int_\Omega \nabla u(\tau,x)\cdot\nabla v(x)\,dx\,d\tau = \int_\Omega u_0(x)\,v(x)\,dx + \int_0^t \int_\Omega f(\tau,x)\,v(x)\,dx\,d\tau, \text{ for all } v \in H_0^1(\Omega) \quad ②$$

Galerkin Method

Goal: Approximate the solution $u(t,x)$ by finite sums $u_N(t,x) = \sum_{k=1}^{N} u_k(t)\phi_k(x)$

Principle: The larger N, the better the approximation

Insert u_N into variational formulation ②.
Solve the resulting system of N ordinary differential equations (ODEs).
Show that the solutions of the ODEs converge as $N \to \infty$.
Show that this limit function solves the variational formulation.

Physical Application

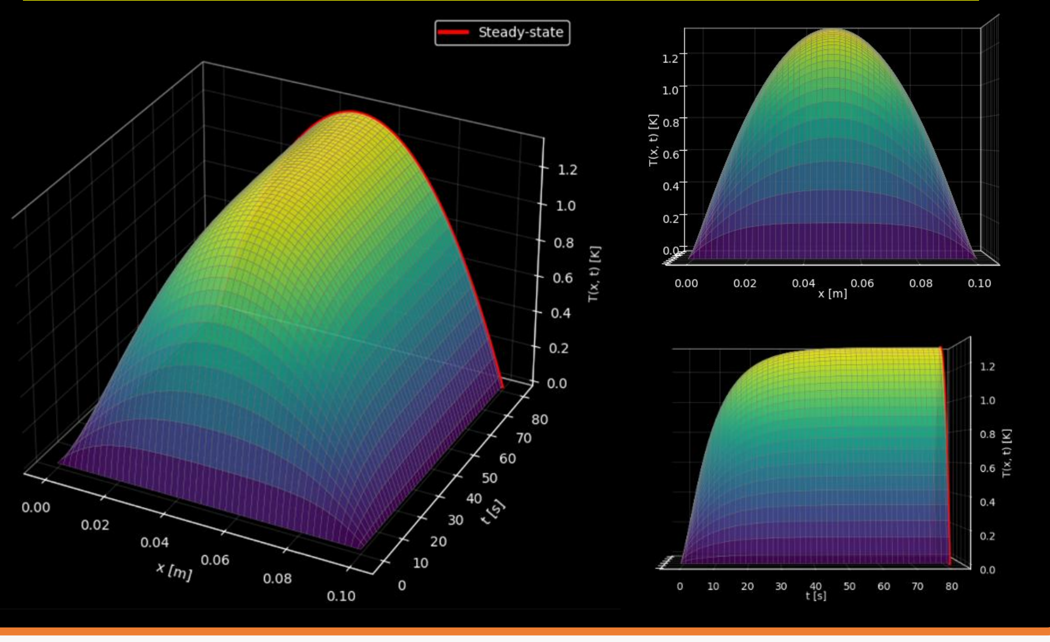
A copper wire of length L carries a constant current that heats it. Initially (t = 0) the wire is at 0 °C, and both ends are kept at 0 °C throughout heating. Given its resistivity ρ_e, cross-section A, thermal conductivity k, density ρ, and heat capacity c_p, find the temperature T(x,t) along the wire over time.

1. Rate of stored energy	2. Fourier's law	3. Net heat conduction	4. Joule heating
$\rho c_p \dfrac{\partial T}{\partial t}$	$q_x = -k\dfrac{\partial T}{\partial x}$	$\Longrightarrow k\dfrac{\partial^2 T}{\partial x^2}$	$\rho_e \left(\dfrac{I}{A}\right)^2$

$$\Longrightarrow \frac{\partial T}{\partial t} - \alpha\frac{\partial^2 T}{\partial x^2} = f, \quad \text{with } \alpha = \frac{k}{\rho c_p},\ f = \frac{\rho_e\left(\frac{I}{A}\right)^2}{\rho c_p}$$

Solve by Fourier series expansion:

$$T(x,t) = \frac{4fL^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3}\left(1 - e^{-(n\pi/L)^2 \alpha t}\right) \sin\frac{n\pi x}{L}, \text{ and } \tilde{T}(x) = \frac{f}{2}x(L-x)$$



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